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The Navier–Stokes Solution for Laminar Flow Past a Semi-infinite Flat Plate

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SUMMARY

A numerical solution joining Carrier and Lin's solution near the leading edge to the boundary layer solution at large distance of the leading edge is presented. The solution is valid for any Reynolds number. Results are given for the skin friction, the integrated skin friction, the displacement thickness, the pressure along the plate and the velocity ahead of the plate. The asymptotic value of the integrated skin friction agrees very well with the exact value. The displacement thickness is already different from zero for small distances ahead of the plate.

1. Introduction

Boundary layer theory provides an asymptotic solution of the Navier–Stokes equations valid for large values of x, that is far downstream the plate. Near the leading edge the nature of the flow has been analysed by Carrier and Lin [1]. Various attempts have been made to join the leading edge solution to the boundary layer solution but all of these introduce additional assumptions. The best results seem to have been obtained by Dean [3] who makes the assumption that the boundary layer solution is approximately valid everywhere and who uses this solution to evaluate the non-linear convective terms in the Navier–Stokes equations. Davis [2] used a series truncation method in which the stream function is locally expanded in a power series in the x-coordinate. He presents results for a first and a second truncation which strongly support Dean's results.

Higher approximations of boundary layer theory, also valid for large values of x, have been given by Stewartson [12], Goldstein [6], van Dyke [4] and Murray [11]. Stewartson showed that the higher approximations must contain logarithmic terms in order to have exponential decay of vorticity in the y-direction, perpendicular to the plate. His solution contains also arbitrary constants, of which the values are fixed by the behaviour at smaller x, but as long as this solution was not joined to the solution at the leading edge, they could not be determined.

Near the leading edge Lewis and Carrier [10] solved the Oseen approximation of the Navier–Stokes equations, but later work showed that this was not a valid first approximation, see e.g. Davis [2] and Lagerstrom [9, pages 89 and 90]. Imai [7] patched Stewartson's asymptotic solution for large x to the Carrier–Lin solution for small x at $R_x = 1$ (R_x is Reynolds number based on the distance x). Although this gives a better numerical result for the skin friction near the leading edge than the Oseen approximation, it can neither be seen as a satisfactory procedure.

The present paper gives the method by which a solution valid for all x is obtained which is exact except for truncation and discretization errors in the numerical solution of the final set of partial differential equations. The dependent variables which are solved are the discrepancies of stream function and vorticity from the values they would assume according to first-order boundary layer theory. This leads to a well-posed boundary value problem in a quarter infinite plane (using parabolic coordinates). In order to avoid the numerical difficulties connected with infinite regions, a transformation to a rectangle is performed. Since the dependent variables are small quantities (this feature accounts for the success of Dean's results) the truncation and discretization errors in our solution will be very small indeed. The solution is valid for any Reynolds number provided the flow is laminar and incompressible.

Results are presented for the local skin friction and the integrated skin friction which are compared to those of other investigators. Also, the displacement thickness of the boundary layer is calculated for all values of x (including negative values). This had to be based upon a generalized definition since the exponential decay of the vorticity combines with an algebraic approach of the stream function in y-direction to its first-order boundary layer value. The displacement thickness is defined as the distance along which the wall should be displaced in order to obtain without viscosity the same outer potential flow as before. Finally, results for the pressure along the plate are presented.

2. Formulation of the Problem

The Navier–Stokes equations for an incompressible, viscous fluid can be combined into one equation for the stream function ψ, see e.g. van Dyke [4],

$$\frac{\partial(\psi, \Delta\psi)}{\partial(y, x)} = \nu\Delta\Delta\psi \tag{2.1}$$

where ν is the kinematic viscosity, Δ is the Laplacian and the Jacobian

$$\frac{\partial(\psi, \Delta\psi)}{\partial(y, x)} \text{ denotes } \frac{\partial\psi}{\partial y}\frac{\partial\Delta\psi}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial\Delta\psi}{\partial y}.$$

The boundary conditions for the semi-infinite plate are

$$\left.\begin{array}{lll} \psi(x, 0) = 0 & & \\ \psi_y(x, 0) = 0 & \text{for} & x > 0 \\ \psi(x, y) \to U_0 y & \text{for} & x \to -\infty \end{array}\right\} \tag{2.2}$$

In various regions of the field different terms of eq. (2.1) can be neglected. Outside the boundary layer viscosity can be neglected and (2.1) becomes

$$\Delta\psi = 0\,.$$

Boundary layer theory is based upon the assumption that in the boundary layer the operator Δ can be replaced by $\partial^2/\partial y^2$, thus neglecting $\partial^2/\partial x^2$. By aid of the transformations

$$x = x_1 l\,, \quad y = y_1\sqrt{\frac{\nu l}{U_0}}\,, \quad \psi = \Psi\sqrt{U_0 \nu l}\,,$$

where l is some reference length and U_0 is the undisturbed speed, eq. (2.1) becomes

$$\frac{\partial(\Psi, \partial^2\Psi/\partial y_1^2)}{\partial(y_1, x_1)} = \frac{\partial^4\Psi}{\partial y_1^4}.$$

After an integration to y_1 the usual boundary layer equation is obtained.

Near the leading edge boundary layer theory becomes inconsistent since it yields a vertical velocity v which becomes infinite as $O(x^{-\frac{1}{2}})$ for $x \to 0$ and hence $\partial v/\partial x = -\partial^2\psi/\partial x^2$ could not be neglected there.

Near the leading edge all terms of eq. (2.1) are equally important and we introduce the following transformation in agreement with Carrier and Lin [1]

$$x = \frac{l x_1}{R}\,, \quad y = \frac{l y_1}{R}\,, \quad \psi = \Psi\nu \tag{2.3}$$

where $R = U_0 l/\nu$ is the Reynolds number based upon the reference length l. x_1 and y_1 are dimensionless coordinates. If these are $O(1)$, which means x/l and y/l are $O(R^{-1})$, the full equation

$$\frac{\partial(\Psi, \Delta\Psi)}{\partial(y_1, x_1)} = \Delta\Delta\Psi, \quad \text{where} \quad \Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial y_1^2}, \tag{2.4}$$

must be taken into account.

It is preferable to write eq. (2.4) as a set of two partial differential equations

$$\begin{aligned} &\frac{\partial(\Psi, \Gamma)}{\partial(y_1, x_1)} = \Delta\Gamma \\ &\Gamma = \Delta\Psi\,. \end{aligned} \tag{2.5}$$

The new quantity Γ is related to the local vorticity ω by the relation

$$\Gamma = \frac{v}{U_0^2}\,\omega, \text{ where } \omega = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}. \tag{2.6}$$

The boundary conditions become

$$\left.\begin{array}{llll} x_1 < 0, & y_1 = 0 & \Psi = 0, & \Gamma = 0 \\ x_1 > 0, & y_1 = 0 & \Psi = 0, & \dfrac{\partial\Psi}{\partial y_1} = 0 \\ x_1 \to -\infty & & \Psi \to y_1, & \Gamma \to 0 \\ y_1 \to \infty & & \dfrac{\partial\Psi}{\partial y_1} \to 1, & \Gamma \to 0 \end{array}\right\} \tag{2.7}$$

The condition $\Gamma = 0$ ahead of the plate ($x_1 < 0$, $y_1 = 0$) follows from reasons of symmetry.

The solution following from eqs. (2.5) and (2.7) should be such that it matches smoothly with the boundary layer solution for $x_1 \to \infty$, $y_1 \leqq O(\sqrt{x_1})$ as well as with the potential theoretical solution outside the boundary layer.

3. Introduction of Parabolic Coordinates

It has first been shown by Kaplun [8] that parabolic coordinates are optimal for the semi-infinite flat plate; this means that the external potential flow is included asymptotically in the boundary layer solution correctly to a higher order (including order $R^{-\frac{1}{2}}$) than when using other coordinates, see also Goldstein [6] and van Dyke [4]. When the full Navier–Stokes equations are used, parabolic coordinates are also preferable. Therefore, we introduce

$$\begin{aligned} x_1 &= \xi^2 - \eta^2 \\ y_1 &= 2\xi\eta \end{aligned} \tag{3.1}$$

Transformation of eqs. (2.5) to the parabolic coordinates ξ, η yields

$$\left.\begin{aligned} &\Delta\Gamma = \frac{\partial(\Psi, \Gamma)}{\partial(\eta, \xi)} \\ &\text{where } \Delta = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}. \\ &\Delta\Psi = 4(\xi^2 + \eta^2)\Gamma \end{aligned}\right\} \tag{3.2}$$

The boundary conditions are

$$\left.\begin{array}{llll} \text{ahead of the plate} & \xi = 0 & \Psi = 0, & \Gamma = 0 \\ \text{at the plate} & \eta = 0 & \Psi = 0, & \dfrac{\partial\Psi}{\partial\eta} = 0 \end{array}\right\} \tag{3.3}$$

$$\left.\begin{array}{llll} \text{boundary layer of plate} & \xi \to \infty & \Psi \to \xi f(2\eta), & \Gamma \to \dfrac{\xi}{\xi^2+\eta^2} f''(2\eta) \\ \text{outside the boundary layer} & \eta \to \infty & \dfrac{\partial \Psi}{\partial \eta} \to 2\xi, & \Gamma \to 0\,. \end{array}\right\} \quad \begin{array}{c}(3.3)\\ \text{(cont'd)}\end{array}$$

In parabolic coordinates the solution for the stream function in the boundary-layer approximation is, Goldstein [6],

$$\Psi = \xi f(2\eta)$$

and this must then be the limiting form to which Ψ approaches in the present theory if $\xi \to \infty$. The pertaining value of Γ follows from the second equation of (3.2). The symbol f denotes the Blasius function defined by

$$2f''' + ff'' = 0, \quad f(0) = f'(0) = 0, \quad f'(\infty) = 1 \tag{3.4}$$

where primes denote differentiations to the argument 2η.

It appears that it is preferable to replace the dependent variable Γ by K, defined by

$$K = (\xi^2 + \eta^2)\Gamma\,. \tag{3.5}$$

While Γ is directly proportional to the local vorticity, we shall denote K by "modified voriticity".

The reason is that near the leading edge of the plate ($\xi = 0$, $\eta = 0$) the solution of Carrier and Lin [1] applies which gives the following behaviour for Ψ and Γ near the origin

$$\Psi \sim 4\xi\eta^2\,, \qquad \Gamma \sim \frac{2\xi}{\xi^2+\eta^2}\,,$$

apart from an unknown multiplicative constant. This shows that Γ is singular near the origin, but that K remains bounded.

The differential equations for Ψ and K become

$$\left.\begin{aligned} &\Delta\left(\frac{K}{\xi^2+\eta^2}\right) = \frac{\partial\{\Psi, K/(\xi^2+\eta^2)\}}{\partial(\eta, \xi)} \\ &\Delta\Psi = 4K, \quad \text{where} \quad \Delta = \frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2}\,. \end{aligned}\right. \tag{3.6}$$

The boundary conditions are

$$\left.\begin{array}{lll} \xi = 0 & \Psi = 0, & K = 0 \\ \eta = 0 & \Psi = 0, & \dfrac{\partial\Psi}{\partial\eta} = 0 \\ \xi \to \infty & \Psi \to \xi f(2\eta), & K \to \xi f''(2\eta) \\ \eta \to \infty & \dfrac{\partial\Psi}{\partial\eta} \to 2\xi, & K \to 0, \text{ see Sect. 4.} \end{array}\right\} \tag{3.7}$$

4. Asymptotic Behaviour for Large Values of the Coordinates

In order to investigate the case $\xi \to \infty$ and/or $\eta \to \infty$ we transform to polar coordinates in the plane of the parabolic variables ξ and η. Hence

$$\xi = r\cos\theta\,, \qquad \eta = r\sin\theta\,. \tag{4.1}$$

After transformation the differential equations become

$$\Delta\left(\frac{K}{r^2}\right) = \frac{1}{r}\,\frac{\partial(\Psi, K/r^2)}{\partial(\theta, r)} \tag{4.2}$$

$$\Delta\Psi = 4K, \quad \text{where} \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\,.$$

For large values of r the boundary layer shrinks to $\theta=0$, since it is well-known that the argument of the Blasius function, 2η, remains finite in the boundary layer. With $r\to\infty$, this implies $\theta\to 0$. Hence, if $r\to\infty$, all non-zero values of θ correspond to potential flow. It can also be said that θ is the outer expansion variable. The boundary layer for large values of r, that is of ξ, can only be investigated from eqs. (3.6) where η is the inner expansion variable.

The main contribution to Ψ for large r comes from the potential flow as modified by the displacement thickness of the boundary layer. This yields

$$\Psi = 2\xi\eta - \beta\xi\,, \tag{4.3}$$

where β is determined from the condition that

$$f(2\eta) \sim 2\eta - \beta \quad \text{for} \quad \eta\to\infty, \quad \beta = 1.72078765\,.$$

Eqn. (4.3) is not the exact solution for the potential flow about plate + displacement thickness, but it gives the first terms of an asymptotic series. Important is that there is no term of $O(1)$ for ξ or $\eta\to\infty$. This is shown in [14]. It corresponds to the fact that in the outer expansions given by Goldstein [5] and van Dyke [4] as

$$\psi = y - \frac{\beta\xi}{R^{\frac{1}{2}}} + \dots \text{ (in their notation)},$$

there is no term of order R^{-1}.

The result (4.3) might, in principle, also be modified by the fact that the Navier–Stokes solution changes the displacement thickness. This, however, is only a local effect, see [14], which neither contributes a term $O(1)$ in (4.3).

Hence, the asymptotic series for Ψ and K begin like

$$\left.\begin{aligned} \Psi &= r^2\sin 2\theta - \beta r\cos\theta + o(1) \\ K &= o(1) \end{aligned}\right\} \quad \text{for } r\to\infty,\ \theta>0. \tag{4.4}$$

In the boundary layer the asymptotic series for large ξ are

$$\Psi = \xi f(2\eta) + o(1), \qquad K = \xi f''(2\eta) + o(1), \ \text{ for } \ \xi\to\infty. \tag{4.5}$$

At the outer side of the boundary layer ($\eta\to\infty$) the solutions (4.4) and (4.5) match.

An important point is that the vorticity Γ and also K decreases exponentially for large values of η. A mathematically rigorous proof of this property is difficult to give, but there exists ample evidence for its truth. Substitution of (4.3) in the first equation (3.2) or (3.6) leads to an equation which only has solutions decreasing exponentially for $\eta\to\infty$. This is shown in [14]. Moreover the numerical results wholly confirmed this behaviour. It should be emphasized that the approach of Ψ to its value (4.3) is algebraic for large η.

5. Transformation to a Rectangular Region

The boundary value problem as formulated by (3.6) and (3.7) has two complications with regard to its numerical solution:

(i) the region is not bounded, being a quarter-infinite plane

(ii) the functions Ψ and K are not always bounded if the coordinates approach infinity.

The second complication can be removed quite easily by introducing as new dependent variables

$$\Psi_1 = \Psi - \xi f(2\eta)\,, \qquad K_1 = K - \xi f''(2\eta)\,. \tag{5.1}$$

The functions Ψ_1 and K_1 denote the differences of stream function and modified vorticity from the values, they assume in first-order boundary-layer theory. Introduction of (5.1) in eqs. (3.6) leads to the new equations

$$\Delta K_1 = \frac{\partial \Psi_1}{\partial \eta}\frac{\partial K_1}{\partial \xi} - \frac{\partial \Psi_1}{\partial \xi}\frac{\partial K_1}{\partial \eta} + \frac{\partial \Psi_1}{\partial \xi}\left\{\xi f f'' + \frac{2\eta(K_1+\xi f'')}{\xi^2+\eta^2}\right\} + \frac{\partial \Psi_1}{\partial \eta}\left\{f'' - \frac{2\xi(K_1+\xi f'')}{\xi^2+\eta^2}\right\} +$$

$$+ \frac{\partial K_1}{\partial \xi}\left(2\xi f' + \frac{4\xi}{\xi^2+\eta^2}\right) + \frac{\partial K_1}{\partial \eta}\left(\frac{4\eta}{\xi^2+\eta^2} - f\right) + \frac{2K_1}{\xi^2+\eta^2}(\eta f - 2\xi^2 f' - 2) + \frac{2\xi\eta f''}{\xi^2+\eta^2}(2\eta f' - f)\,.$$

$$\Delta \Psi_1 = 4K_1, \quad \text{where} \quad \Delta = \frac{\partial^2}{\partial \xi^2} + \frac{\partial^2}{\partial \eta^2}\,. \tag{5.2}$$

In this derivation use has been made of eq. (3.4).

The boundary conditions are now completely homogeneous, viz.

$$\left.\begin{array}{lll} \xi=0 & \Psi_1=0 & K_1=0 \\ \eta=0 & \Psi_1=0 & \dfrac{\partial \Psi_1}{\partial \eta}=0 \\ \xi\to\infty & \Psi_1\to 0 & K_1\to 0 \\ \eta\to\infty & \Psi_1\to 0 & K_1\to 0 \end{array}\right\} \tag{5.3}$$

The condition for $\eta\to\infty$ has been obtained by using (4.3) and remarking that this is the asymptotic expansion of $\xi f(2\eta)$ for $\eta\to\infty$, if exponentially small terms are neglected.

There is a non-trivial solution of the boundary value problem since the equation for K_1 is not homogeneous.

The first difficulty (i) is removed by transforming the quarter-infinite plane to a rectangle. Considerable care has to be taken in this step. The transformation should be such that the derivatives of Ψ_1 and K_1 to the new independent variables remain finite since otherwise the accuracy of the numerical solution may be badly affected. This means that the transformation depends upon the way in which Ψ_1 and K_1 approach zero.

We first consider the transformation in ξ-direction. Stewartson [12] has shown that we must have for $\xi\to\infty$

$$\begin{aligned} \Psi_1 &\sim \frac{\ln \xi}{\xi} h(2\eta) + \frac{1}{\xi} g(2\eta) \\ K_1 &\sim \frac{\ln \xi}{\xi} h''(2\eta) + \frac{1}{\xi} g''(2\eta) \end{aligned} \tag{5.4}$$

in order to obtain exponential decrease of K_1 for large values of η. The relations (5.4) are such that the second equation (5.2) is automatically satisfied, to the order considered here. Substitution of (5.4) into the first equation (5.2) leads to equations for h and g, see Appendix.

The transformation we introduce is

$$\sigma = 1 - \frac{\ln(1+\xi/2)}{\xi/2}\,. \tag{5.5}$$

Near $\xi=0$ the transformation behaves as $\sigma=\frac{1}{4}\xi$, while for $\xi\to\infty$ we have

$$\sigma \sim 1 - \frac{\ln(\xi/2)}{\xi/2}\,.$$

The variable ξ has been multiplied by $\frac{1}{2}$ in order to have at the middle of the σ-interval, namely at $\sigma=0.5$, a value of ξ equal to 5, to which correspond maximum values of Ψ_1 and K_1 (this follows from the numerical calculations).

It follows from eqs. (5.4) and (5.5) that $\partial\Psi_1/\partial\sigma$ and $\partial K_1/\partial\sigma$ remain finite for $\xi\to\infty$ $(\sigma\to 1)$. For example, for $\xi\to\infty$,

$$\frac{\partial\Psi_1}{\partial\sigma}=\frac{\partial\Psi_1}{\partial\xi}\frac{d\xi}{d\sigma}=\left\{-\frac{\ln\xi}{\xi^2}h+\frac{1}{\xi^2}(h-g)\right\}\Big/\left\{\frac{\ln(\xi/2)}{\xi^2/2}-\frac{2}{\xi^2}\right\}=-\tfrac{1}{2}h+O\left(\frac{1}{\ln\xi}\right).$$

Eqs. (5.4) give the beginning of the asymptotic series for large ξ in the boundary layer, comparable to eq. (4.5). We now want to find the equivalent of eq. (4.4), that is the asymptotic series in the remaining part of the r, θ-plane. It has been observed in Sect. 4 that K decreases exponentially and hence Ψ_1 will be a harmonic function outside the boundary layer.

The harmonic function Ψ_1 should match to the value

$$\Psi_1\approx\frac{\ln\xi}{\xi}h(\infty)+\frac{1}{\xi}g(\infty)\ \text{at}\ \theta=0\,. \tag{5.6}$$

For matching the first term we need a harmonic function which behaves like $\xi^{-1}\ln\xi$ along the ξ-axis. We remark that the real part of a holomorphic function in the complex plane is a harmonic function. Therefore we take for the first term

$$\operatorname{Re}\frac{\ln z}{z}=\cos\theta\cdot\frac{\ln r}{r}+\frac{\theta\sin\theta}{r}.$$

However, since the harmonic function should vanish along the η-axis ($\theta=\pi/2$), we modify it to

$$\cos\theta\cdot\frac{\ln r}{r}-\left(\frac{\pi}{2}-\theta\right)\frac{\sin\theta}{r}$$

since $r^{-1}\sin\theta$ is also a harmonic function.

The second term of (5.6) gives rise to the harmonic function $r^{-1}\cos\theta$. Hence, the total harmonic function matching with (5.6) for $\theta=0$, is given by

$$\Psi_1=\left\{\cos\theta\cdot\frac{\ln r}{r}-\left(\frac{\pi}{2}-\theta\right)\frac{\sin\theta}{r}\right\}h(\infty)+\frac{\cos\theta}{r}g(\infty)\,. \tag{5.7}$$

In ξ, η coordinates this becomes (outside the boundary layer)

$$\Psi_1=\left\{\frac{\frac{1}{2}\xi\ln(\xi^2+\eta^2)}{\xi^2+\eta^2}-\frac{\eta}{\xi^2+\eta^2}\tan^{-1}\frac{\xi}{\eta}\right\}h(\infty)+\frac{\xi}{\xi^2+\eta^2}g(\infty)\,. \tag{5.8}$$

For large η and finite ξ this contains terms of order

$$\frac{\ln\eta}{\eta^2}\quad\text{and}\quad\frac{1}{\eta^2}.$$

The transformation in η-direction, which has been applied is

$$\eta=\frac{5\tau}{5-4\tau}+5\tau^2(1-\tau^2)\,. \tag{5.9}$$

The $[0,\infty]$ interval in η is transformed to the $[0, 1.25]$ interval in τ.

The transformation (5.9) makes $\partial\Psi_1/\partial\tau$ equal to 0 for $\tau=1.25$. For $\eta=0$ the derivative $d\eta/d\tau$ is equal to 1. The specific form of the transformation (5.9) is determined for a great part by requirements of stability of the numerical solution, see Sect. 6.

Applying the transformations (5.5) and (5.9) the partial differential equations become

$$\left(\frac{d\sigma}{d\xi}\right)^2 \frac{\partial^2 K_1}{\partial\sigma^2} + \left(\frac{d\tau}{d\eta}\right)^2 \frac{\partial^2 K_1}{\partial\tau^2} = \left(\frac{\partial\Psi_1}{\partial\tau}\frac{\partial K_1}{\partial\sigma} - \frac{\partial\Psi_1}{\partial\sigma}\frac{\partial K_1}{\partial\tau}\right)\frac{d\sigma}{d\xi}\frac{d\tau}{d\eta} \tag{5.10}$$

$$+ \frac{\partial\Psi_1}{\partial\sigma}\frac{d\sigma}{d\xi}\left\{\xi f f'' + \frac{2\eta(K_1+\xi f'')}{\xi^2+\eta^2}\right\} + \frac{\partial\Psi_1}{\partial\tau}\frac{d\tau}{d\eta}\left\{f'' - \frac{2\xi(K_1+\xi f'')}{\xi^2+\eta^2}\right\} +$$

$$+ \frac{\partial K_1}{\partial\sigma}\left\{\frac{d\sigma}{d\xi}\left(2\xi f' + \frac{4\xi}{\xi^2+\eta^2}\right) - \frac{d^2\sigma}{d\xi^2}\right\} + \frac{\partial K_1}{\partial\tau}\left\{\frac{d\tau}{d\eta}\left(\frac{4\eta}{\xi^2+\eta^2} - f\right) - \frac{d^2\tau}{d\eta^2}\right\} +$$

$$+ \frac{2K_1}{\xi^2+\eta^2}(\eta f - 2\xi^2 f' - 2) + \frac{2\xi\eta f''}{\xi^2+\eta^2}(2\eta f' - f)\,.$$

$$\left(\frac{d\sigma}{d\xi}\right)^2 \frac{\partial^2 \Psi_1}{\partial\sigma^2} + \left(\frac{d\tau}{d\eta}\right)^2 \frac{\partial^2 \Psi_1}{\partial\tau^2} + \frac{d^2\sigma}{d\xi^2}\frac{\partial\Psi_1}{\partial\sigma} + \frac{d^2\tau}{d\eta^2}\frac{\partial\Psi_1}{\partial\tau} = 4K_1\,. \tag{5.10}$$

The boundary conditions become

$$\left.\begin{array}{lll} \sigma=0 & \Psi_1=0, & K_1=0 \\ \tau=0 & \Psi_1=0, & \dfrac{\partial\Psi_1}{\partial\tau}=0 \\ \sigma=1 & \Psi_1=0, & K_1=0 \\ \tau=1.25 & \Psi_1=0, & K_1=0\,. \end{array}\right\} \tag{5.11}$$

6. The Numerical Solution

Eqs. (5.10) have been replaced by a system of difference equations, based on a net of which the netpoints are $\sigma=0(1/32)1$ and $\tau=0(1/32)1(1/128)1.25$. Derivatives have been replaced by simple difference expressions such as

$$\frac{\partial K_1}{\partial\sigma}(\sigma,\tau) = \frac{K_1(\sigma+h,\tau)-K_1(\sigma-h,\tau)}{2h}$$

$$\frac{\partial^2 K_1}{\partial\sigma^2}(\sigma,\tau) = \frac{K_1(\sigma+h,\tau)-2K_1(\sigma,\tau)+K_1(\sigma-h,\tau)}{h^2}$$

where h is the step in σ-direction. For the derivatives to τ at $\tau=1$, where the step in τ-direction changes in magnitude, special formulae have been derived in an obvious way.

A slight complication forms the second boundary condition (5.11) since there $\partial\Psi_1/\partial\tau$ instead of K_1 vanishes. The value of K_1 at $\tau=0$ then follows from the second equation (5.10) which for $\tau=0$ simplifies to ($d\tau/d\eta=1$)

$$\frac{\partial^2\Psi_1}{\partial\tau^2} = 4K_1\,.$$

Taking into account that Ψ_1 and $\partial\Psi_1/\partial\tau$ vanish for $\tau=0$, the difference form of this equation becomes

$$K_1^{(n+1)}(\sigma,0) = \frac{\Psi_1^{(n)}(\sigma,h)}{2h^2}\,.$$

The system of difference equations is solved by iteration, taking e.g. the $(n+1)^{\text{th}}$ approximate value $K_1^{(n+1)}(\sigma,\tau)$ from the neighbouring values of $K_1^{(n)}(\sigma\pm h,\tau)$, $K_1^{(n)}(\sigma,\tau\pm k)$ and similar values of $\Psi_1^{(n)}$. However, it has already been found by Thom and Apelt [13], their experience being confirmed by our calculations, that this iteration procedure was nonstable. It is necessary to accept as new value for the $(n+1)^{\text{th}}$ step not the value given by the difference equation, but to introduce an underrelaxation factor ω yielding as new value for the $(n+1)^{\text{th}}$ step

$$K_1^{(n)}(\sigma,\tau)+\omega\{K_1^{(n+1)}(\sigma,\tau)-K_1^{(n)}(\sigma,\tau)\}\,. \tag{6.1}$$

It was found that with $\omega=0.5$ the iteration converged (slowly), but that larger values often gave rise to divergences. The underrelaxation factor is applied to both K_1 and Ψ_1.

The reason of the instability lies in the first equation (5.10). The following analysis shows that it may happen that the error in $K^{(n+1)}(\sigma,\tau)$ is larger than that in $K^{(n)}(\sigma,\tau)$. Considering in the difference equation arising from the first equation (5.10) only the terms referring to the point (σ,τ),

$$-\frac{2\left\{\left(\frac{d\sigma}{d\xi}\right)^2+\left(\frac{d\tau}{d\eta}\right)^2\right\}}{h^2}K_1^{(n+1)}(\sigma,\tau)=\frac{2}{\xi^2+\eta^2}\left\{\eta\frac{\partial\Psi_1}{\partial\xi}-\xi\frac{\partial\Psi_1}{\partial\eta}+(\eta f-2\xi^2f'-2)\right\}\times$$
$$\times K_1^{(n)}(\sigma,\tau)+\dots$$

or

$$K_1^{(n+1)}(\sigma,\tau)=\frac{h^2}{\xi^2+\eta^2}\cdot\frac{\xi\frac{\partial\Psi_1}{\partial\eta}-\eta\frac{\partial\Psi_1}{\partial\xi}+(2+2\xi^2f'-\eta f)}{\left(\frac{d\sigma}{d\xi}\right)^2+\left(\frac{d\tau}{d\eta}\right)^2}\cdot K_1^{(n)}(\sigma,\tau)+\dots$$

which we write as

$$K_1^{(n+1)}(\sigma,\tau)=(Q+1)K_1^{(n)}(\sigma,\tau)+\dots.$$

Applying now eq. (6.1), we see that the new value for the $(n+1)^{\text{th}}$ step in the point (σ,τ) becomes

$$(\omega Q+1)K_1^{(n)}(\sigma,\tau)\,.$$

It will be clear that if $K_1^{(n)}(\sigma,\tau)$ possesses an error ε, the error in the same point at the next step becomes

$$(\omega Q+1)\varepsilon\,.$$

For stability we should have

$$-2<\omega Q<0\,. \tag{6.2}$$

The largest values of Q arise for $\xi\to\infty$, $\eta\to\infty$. Since $f'\to1$ for $\eta\to\infty$, we can write approximately

$$Q+1\sim\frac{2h^2}{\left(\frac{d\sigma}{d\xi}\right)^2+\left(\frac{d\tau}{d\eta}\right)^2},\quad\text{if}\quad \eta \text{ is large and } \xi\to\infty\,.$$

The difficulty is that for large values of ξ and η the denominator becomes very small leading to positive values of Q, for which the stability criterion (6.2) never can be satisfied. In order to make Q negative, we should have

$$\left(\frac{d\sigma}{d\xi}\right)^2+\left(\frac{d\tau}{d\eta}\right)^2>2h^2\,. \tag{6.3}$$

This requirement has partly determined the transformation (5.9). Since K decreases exponentially in η-direction, it appeared that we can safely put $K=0$ if $\eta\geqq5$. For these values of η, the equation for K can be left out of account. We only have to take care of the stability of the calculations for $\eta<5$, corresponding to $\tau<1$. We wanted stability for both $h=\frac{1}{16}$ and $h=\frac{1}{32}$ since this enables us to estimate the discretization error by comparison of the two calculations. Since for large values of ξ, $d\sigma/d\xi$ vanishes, the final requirement is that at the points with the highest τ-values where the difference equation is applied (that is at $\tau=\frac{15}{16}$), we should have

$$\frac{d\tau}{d\eta} > \frac{\sqrt{2}}{16}.$$

The transformation (5.9) satisfies this requirement and gives no stability difficulties.

It may finally be remarked that the values of ξ corresponding to the values of σ given by $0(h)1$ are found from eq. (5.5) by aid of a Newton–Raphson procedure.

7. The Displacement Thickness

The displacement thickness is defined as the distance along which the wall should be displaced in order to obtain without viscosity the same outer potential flow as with the original wall and viscosity. According to eq. (5.1) we have

$$\Psi = \Psi_1 + \xi f(2\eta).$$

Outside the boundary layer the last term is equal to the harmonic function $\xi(2\eta-\beta)$. It was found in the preceding section that for $\eta>5$ Ψ_1 is also harmonic. When we denote the analytic continuation of Ψ_1 from the region $\eta>5$ into the region $0<\eta<5$ by Ψ_0, the curve

$$\Psi_0(\xi,\eta)+\xi(2\eta-\beta)=0 \tag{7.1}$$

gives the displaced wall since this corresponds to $\Psi=0$ if there is no viscosity and the outer flow is the same.

The difficulty is to find the analytic continuation Ψ_0. This has to satisfy the following conditions

Ψ_0 and $\partial\Psi_0/\partial\eta$ are given for $\eta=5$

$\Psi_0=0$ for $\xi=0$

$\Psi_0\to 0$ for $\xi\to\infty$

We want to determine Ψ_0 in the semi-infinite strip $\xi\geqq 0$, $0\leqq\eta\leqq 5$ since the displaced wall lies certainly herein. This cannot be done by a numerical method calculating Ψ_0 by aid of differences from the equation $\Delta\Psi_0=0$ at lines $\eta=$constant from $\eta=5$ to $\eta=0$, since this is a highly unstable method. The problem has been solved by aid of Green's function $G^{(1)}$ of the first kind.

Assume that Ψ_0 is known at every point Q of the strip boundary C, then Ψ_0 in an arbitrary point P of the strip follows from

$$\Psi_0(P)=\oint_C \Psi_0(Q)\frac{\partial G^{(1)}}{\partial n}(P,Q)ds_Q, \tag{7.2}$$

where n is the normal to C, directed outward.

In particular, we can calculate by aid of this formula $\partial\Psi_0/\partial\eta$ at the upper side $\eta=5$. But, in reality, this function has been given, while Ψ_0 at the lower side $\eta=0$ is unknown. Hence, we obtain an integral equation for Ψ_0 along the line $\eta=0$. Once Ψ_0 has been solved we can calculate Ψ_0 in any point of the strip by the usual difference method for solving Dirichlet's problem. Then, for each ξ, the value of η can be determined satisfying the relation (7.1). This gives the displacement thickness.

It is shown in [14] that Green's function of the first kind for the semi-infinite strip

$$\xi\geqq 0,\quad 0\leqq\eta\leqq\eta_0$$

is given by

$$G^{(1)}(P,Q)=\frac{1}{4\pi}\ln\frac{(\operatorname{Cosh}\bar{\xi}\cos\bar{\eta}-\operatorname{Cosh}\bar{\xi}_1\cos\bar{\eta}_1)^2+(\operatorname{Sinh}\bar{\xi}\sin\bar{\eta}-\operatorname{Sinh}\bar{\xi}_1\sin\bar{\eta}_1)^2}{(\operatorname{Cosh}\bar{\xi}\cos\bar{\eta}-\operatorname{Cosh}\bar{\xi}_1\cos\bar{\eta}_1)^2+(\operatorname{Sinh}\bar{\xi}\sin\bar{\eta}+\operatorname{Sinh}\bar{\xi}_1\sin\bar{\eta}_1)^2}$$

where $\bar{\xi}=\pi\xi/\eta_0$, $\bar{\xi}_1=\pi\xi_1/\eta_0$, $\bar{\eta}=\pi\eta/\eta_0$, $\bar{\eta}_1=\pi\eta_1/\eta_0$, while ξ, η and ξ_1, η_1 are the coordinates of P and Q, respectively.

The contribution of the lower side $\eta=0$ to the normal derivative at the upper side $\eta=\eta_0$ is given by

$$\left\{\frac{\partial\Psi_0}{\partial\eta}(\xi,\eta_0)\right\}_{\text{lower}} = -\int_0^\infty \Psi_0(\xi_1,0)\left(\frac{\partial^2 G^{(1)}}{\partial\eta\,\partial\eta_1}\right)_{\substack{\eta_1=\eta_0\\ \eta=0}} d\xi_1\,.$$

After substitution of (7.3) we find

$$\left\{\frac{\partial\Psi_0}{\partial\eta}(\xi,\eta_0)\right\}_{\text{lower}} = -\frac{1}{\eta_0}\int_0^\infty \Psi_0(\xi_1,0)\frac{\operatorname{Sinh}\xi\,\operatorname{Sinh}\xi_1}{(\operatorname{Cosh}\xi+\operatorname{Cosh}\xi_1)^2}d\xi_1\,. \tag{7.4}$$

Similarly, the contribution of the upper side to the normal derivative at the upper side is equal to

$$\left\{\frac{\partial\Psi_0}{\partial\eta}(\xi,\eta_0)\right\}_{\text{upper}} = -\frac{\sinh\xi}{\pi}\int_0^\infty \frac{\Psi_0'(\xi_1)-\Psi_0'(\xi)}{\operatorname{Cosh}\xi_1-\operatorname{Cosh}\xi}d\xi_1+\frac{1}{\pi}\xi\Psi_0'(\xi) \tag{7.5}$$

where Ψ_0' stands for $\partial\Psi_0/\partial\xi$ along $\eta=\eta_0$. The derivation of this last formula is given in [14].

The sum of eqs. (7.4) and (7.5) leads to an integral equation of the first kind with $\Psi_0(\xi_1, 0)$ as unknown function. Solution of this integral equation by collocation yields a set of algebraic equations which is ill-conditioned. The reason for this is that oscillations in $\Psi_0(\xi, 0)$ have only small influence on $\partial\Psi_0/\partial\eta$ in points (ξ, η_0). Therefore, these oscillations always creep into a solution obtained by collocation.

A good approximate solution has been obtained by transforming the integral equation to the variable σ defined by eq. (5.5). A Galerkin type of approach was followed by approximating the unknown function as a finite Fourier series and equating Fourier coefficients of both sides of the integral equation. Let the original integral equation be

$$-\frac{1}{\eta_0}\int_0^\infty \Psi_0(\xi_1,0)\frac{\operatorname{Sinh}\xi\,\operatorname{Sinh}\xi_1}{(\operatorname{Cosh}\xi+\operatorname{Cosh}\xi_1)^2}\,d\xi_1=R(\xi)$$

After transformation to the coordinate σ, this becomes

$$-\frac{\pi}{\eta_0^2}\int_0^1 \Psi_0(\sigma_1,0)\frac{\operatorname{Sinh}\xi\,\operatorname{Sinh}\xi_1}{(\operatorname{Cosh}\xi+\operatorname{Cosh}\xi_1)^2}\frac{d\xi_1}{d\sigma_1}\,d\sigma_1=R(\sigma)\,.$$

Since Ψ vanishes for $\sigma_1=0$ and $\sigma_1=1$, we put

$$-\frac{\pi}{\eta_0^2}\Psi_0(\sigma_1,0)=\sum_{j=1}^{n} c_j\sin j\pi\sigma_1\,.$$

Then $\displaystyle\sum_{j=1}^{n} c_j\operatorname{Sinh}\xi\int_0^1 \sin j\pi\sigma_1\frac{\operatorname{Sinh}\xi_1}{(\operatorname{Cosh}\xi+\operatorname{Cosh}\xi_1)^2}\frac{d\xi_1}{d\sigma_1}\,d\sigma_1=R(\sigma)\,.$

Both sides of this equation are expanded in Fourier sine series and the first n coefficients are equated. This yields the set of equations

$$\sum_{j=1}^{n} c_j\int_0^1 \sin i\pi\sigma\,\operatorname{Sinh}\xi\,d\sigma\int_0^1 \sin j\pi\sigma_1\frac{\operatorname{Sinh}\xi_1}{(\operatorname{Cosh}\xi+\operatorname{Cosh}\xi_1)^2}\frac{d\xi_1}{d\sigma_1}\,d\sigma_1=\int_0^1 R(\sigma)\sin i\pi\sigma\,d\sigma\,. \tag{7.6}$$

By taking n not too large, e.g. 4, the higher oscillations are avoided. Moreover, even if $\Psi_0(\sigma, 0)$ showed a small oscillation e.g. for $n=5$ and 6, this oscillation had completely disappeared for the η-values which followed from eq. (7.1). Therefore, a smooth curve has been obtained for the displacement thickness.

8. Results

8.1. *Skin friction*

The local coefficient of skin friction for one side of the plate is, τ being the shear stress

$$c_f = \frac{\tau}{\frac{1}{2}\rho U_0^2} = \frac{2\nu}{U_0^2}\left.\frac{\partial^2\psi}{\partial y^2}\right|_{y=0} = 2\left.\frac{\partial^2\Psi}{\partial y_1^2}\right|_{y_1=0} = \frac{1}{2\xi^2}\left.\frac{\partial^2\Psi}{\partial\eta^2}\right|_{\eta=0} = \frac{2}{\xi^2}K(\xi, 0)\,.$$

Using eq. (5.1) we obtain

$$c_f = \frac{2}{\xi}f''(0) + \frac{2}{\xi^2}K_1(\xi, 0)\,, \tag{8.1}$$

where $f''(0) = 0.332057336$ is the value given by boundary layer theory and $K_1(\xi, 0)$ is the value coming from the numerical solution of Sect. 6.

The results for

$$\xi c_f = c_f\sqrt{R_x} \text{ as function of } R_x = U_0 x/\nu = \xi^2$$

are given in table 1. They have been compared with the results of others in fig. 1. The number

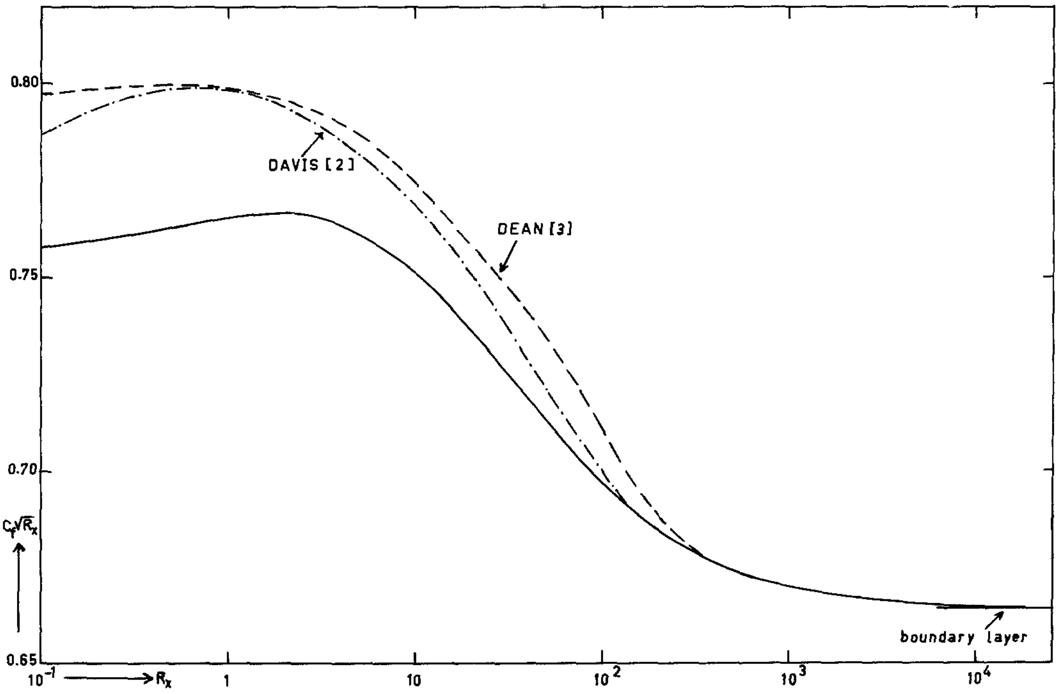

Figure 1. Local skin friction.

$\lim\limits_{R_x \to 0} c_f\sqrt{R_x}$ has been calculated by several investigators and the values are listed below (in chronological order).

Carrier–Lin [1]	0.664 (= boundary layer value)
Lewis–Carrier [10]	1.128 ($= 2/\sqrt{\pi}$, Oseen value)
Dean [3]	0.796
Imai [7]	0.727
Davis [2]	0.779
Present result	0.755.

8.2. *Integrated skin friction*

The integrated skin friction coefficient is given by

$$C_F = \frac{1}{x}\int_0^x c_f\,dx = \frac{4}{\xi}f''(0) + \frac{4}{\xi^2}\int_0^\xi \frac{K_1(\lambda, 0)}{\lambda}\,d\lambda = \frac{1.328}{\xi} + \frac{4}{\xi^2}I\,, \tag{8.2}$$

where the integral I is calculated by transformation to the σ-variable (5.5). Then

$$I = \int_0^\sigma \frac{K_1(\sigma_1, 0)}{1-\sigma_1-2/(2+\xi)}\,d\sigma_1\,, \tag{8.3}$$

where $\xi = \xi(\sigma_1)$.

The numerical results for this integral are given in table 1. They have been compared with the results of other authors in fig. 2.

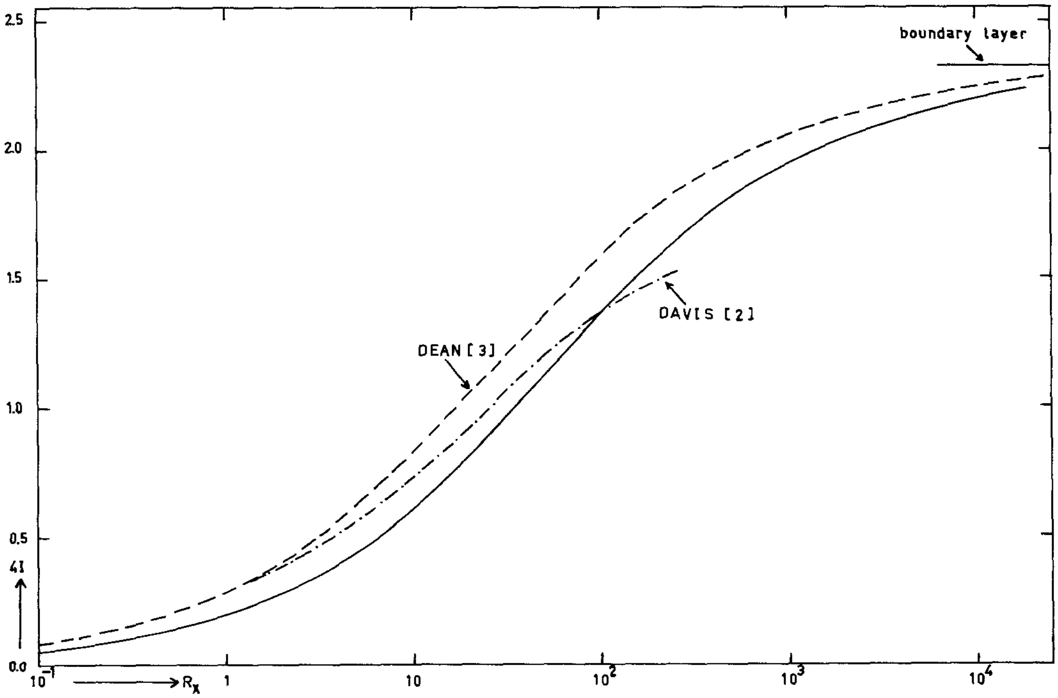

Figure 2. Integrated skin friction: $4I = (C_F - 1.328/\sqrt{R_x})R_x$.

8.3. *Asymptotic behaviour of the integrated skin friction*

The asymptotic expansion of C_F for large values of ξ can be derived from eqs. (8.2) and (5.4). The result is (see also [4] and [7])

$$C_F = 1.328\xi^{-1} + 4I_\infty\xi^{-2} - 4\{h''(0)\cdot(1+\log\xi) - 0.332C_2\}\xi^{-3} + o(\xi^{-3}) \tag{8.4}$$

where $\xi = \sqrt{R_x} \to \infty$.

The value of $h''(0)$ is given in the Appendix as 0.55090346.

The coefficient I_∞ denotes the value of the integral I (8.3) for $\sigma = 1$ or equivalently $\xi = \infty$ (8.2). According to our calculations $4I_\infty = 2.343$, which is the value obtained with $h = \frac{1}{32}$, h being the mesh length in the σ, τ-plane. This result should be compared with the exact value of 2.326 obtained by Imai [7] from momentum considerations, see also van Dyke [4]. Under the assumption that the discretization error is $O(h^2)$ we obtain from the values

$$4I_\infty = 2.407 \quad (h = \tfrac{1}{16})$$
$$4I_\infty = 2.343 \quad (h = \tfrac{1}{32}),$$

the extrapolated result $4I_\infty = 2.322$, which is still closer to Imai's value.

Finally the coefficient C_2 in (8.4) has been considered. This coefficient has been introduced in the Appendix as an unknown multiplier of the first eigenfunction in higher order boundary layer theory. For the determination of C_2 we use eqs. (5.4). The constant C_2 is hidden in the function $g(2\eta)$, see also the Appendix. In the first eq. (5.4) viz.

$$\Psi_1 \sim C_1 \frac{\ln \xi}{\xi} e(2\eta) + \frac{1}{\xi}\{C_2\, e(2\eta) + \Delta g(2\eta)\} \tag{8.5}$$

where $e(2\eta) = f - 2\eta f'$, everything is known except C_2.

The value of C_2 must be such that the asymptotic behaviour matches with $\Psi_1(\xi, \eta)$ coming from Sect. 6, for large values of ξ. We take a large value of ξ and consider eq. (8.5) as an equation for C_2, thus neglecting higher order terms in the expansion. This has been done for several values of η, both inside and outside the boundary layer. In the last region the harmonic continuation (5.8) of (8.5) has been used.

With the result for C_2 the asymptotic behaviour (5.4) of $K_1(\xi, \eta)$ has been checked. The computations have been grouped together in table 2. The result for C_2 is

$$C_2 = -2.0 \pm 10\%.$$

8.4. *Displacement thickness*

The equation (7.1) gives the displacement thickness in terms of the parabolic coordinates ξ and η. The function $\Psi_0(\xi, \eta)$ is the analytic continuation into the region $\eta < 5$ of the function $\Psi_1(\xi, \eta)$. The last function is harmonic for $\eta > 5$.

The numerical values of $\Psi_1(\xi, \eta)$ along the line $\eta = 5$ have been taken from the calculations of Sect. 6. Also the derivatives $\partial\Psi_1/\partial\xi$ and $\partial\Psi_1/\partial\eta$ along $\eta = 5$ have been computed.

At the line $\eta = 5$ the function Ψ_0 must coincide with Ψ_1, while the derivative $\partial\Psi_0/\partial\eta$ has been taken equal to $\partial\Psi_1/\partial\eta$. Then the integral equation for Ψ_0 $(\xi, 0)$ has been set up, see eqs. (7.4)

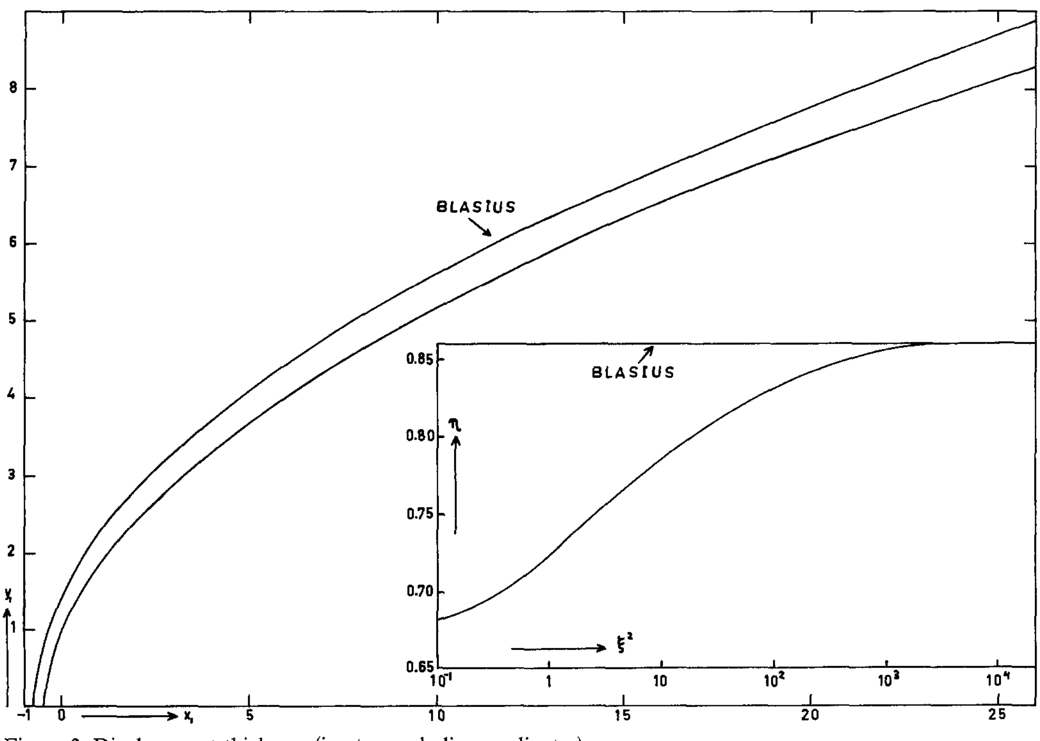

Figure 3. Displacement thickness (inset: parabolic coordinates).

and (7.5). Next the Fourier sine coefficients c_j have been calculated with eqs. (7.6). The function $\Psi_0(\sigma, 0)$ corresponding to $\Psi_0(\xi, 0)$ has been evaluated from its truncated Fourier series, and the Dirichlet problem for Ψ_0 has been solved with the Gauss–Seidel method using an over-relaxation factor of 1.6. This has been performed on the square $(0 \leqq \sigma \leqq 1; 0 \leqq \tau \leqq 1)$ corresponding to the strip $(0 \leqq \xi \leqq \infty; 0 \leqq \eta \leqq 5)$, see eqs. (5.5) and (5.9). Finally the root $\eta = \eta^*$ of (7.1) has been calculated for $\xi = \xi(\sigma)$ where $\sigma = 0(h)1$ and $h = \frac{1}{32}$. Thus

$$\Psi_0(\xi, \eta^*) + \xi(2\eta^* - \beta) = 0, \quad \xi = \xi(\sigma). \tag{8.6}$$

The displacement thickness η^* has been presented in fig. 3. Also the parabola $\eta = \beta/2$ has been given. The latter is the result from boundary-layer theory and may be found from eq. (8.6) by neglecting Ψ_0. For numerical results see table 1. These values correspond to $N = 4$, N denoting the number of Fourier coefficients used above. For $N = 5$ or 6 deviations occur, due to the higher oscillations. It appears that this is a local effect, which is only important for small values of ξ. Therefore the numerical results for the displacement thickness should be considered with some care, since they have not the same accuracy as the other results of this paper.

8.5. *Pressure at the plate and velocity ahead of the plate*

From the Navier–Stokes equation in x-direction, viz.

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\Delta u \tag{8.7}$$

it follows that at the plate

$$\frac{1}{\rho}\frac{\partial p}{\partial x} = \nu\Delta u\,.$$

With $u = \partial\psi/\partial y$ we find, for the pressure at the plate, using (2.3) and (3.1)

$$\frac{\partial \bar{p}}{\partial \xi} = \frac{1}{\xi^2}\frac{\partial K}{\partial \eta}\bigg|_{\eta=0}.$$

The dimensionless pressure $\bar{p}$ is $\bar{p} = p/(\rho U_0^2)$ and K is the modified vorticity (3.5).

Using (5.1) and the fact that $f'''(0) = 0$ we obtain

$$\frac{\partial \bar{p}}{\partial \xi}(\xi, 0) = \frac{1}{\xi^2}\frac{\partial K_1}{\partial \eta}(\xi, 0)$$

or

$$\bar{p}(\xi, 0) = \bar{p}_\infty - \int_\xi^\infty \frac{1}{\lambda^2}\frac{\partial K_1}{\partial \eta}(\lambda, 0)d\lambda\,. \tag{8.8}$$

The integral has been calculated after transformation to the σ-variable (5.5). The value of the normal derivative $\partial K_1/\partial \eta$ has been computed from the values of $K_1(\xi, \eta)$ for $n \geqq 0$ with quadratic interpolation. The results have been checked asymptotically with the approximation

$$\bar{p}(\xi, 0) \sim \bar{p}_\infty + \frac{\beta^2}{8\xi^2} \quad (\xi \to \infty) \tag{8.9}$$

where β is given in (4.3). The approximation (8.9) may be found from (5.4) and the Appendix. For results see table 1 and fig. 4.

Second, the velocity $u(x, 0)$ ahead of the plate has been investigated. This is given by

$$\bar{u}(x_1, 0) = \frac{u}{U_0} = \frac{\partial \Psi}{\partial y_1} = \frac{1}{2\eta}\frac{\partial \Psi}{\partial \xi} = \frac{f(2\eta)}{2\eta} + \frac{1}{2\eta}\frac{\partial \Psi_1}{\partial \xi}(0, \eta) \tag{8.10}$$

where $\xi = y_1 = 0$ and $x_1 = -\eta^2$. The results have been presented in fig. 5.

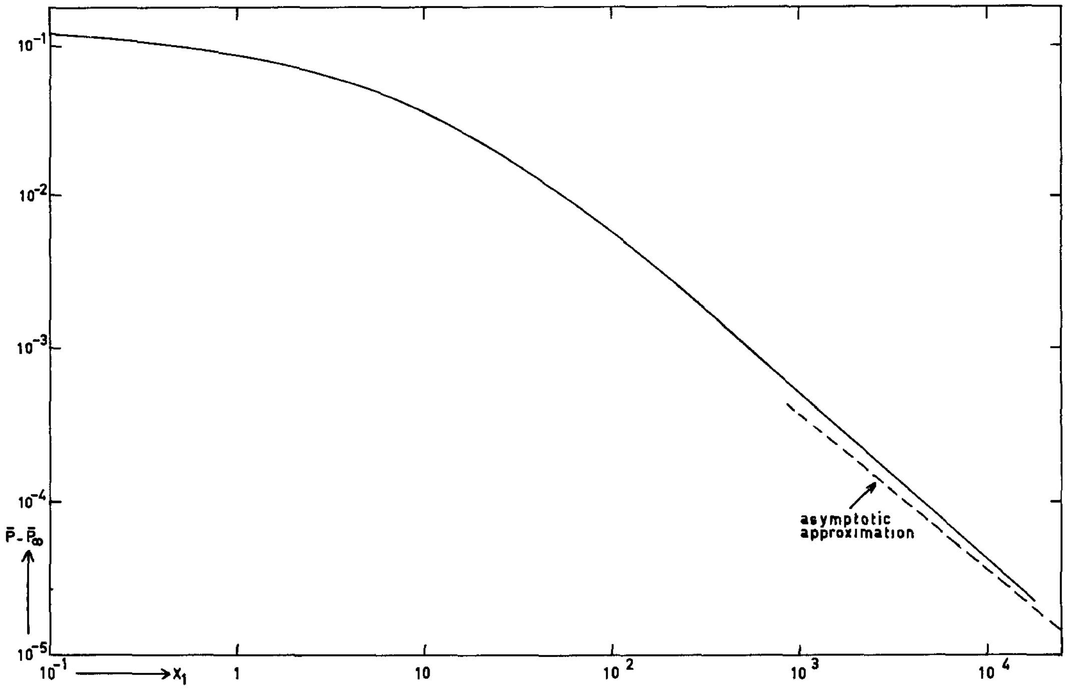

Figure 4. Pressure at the plate.

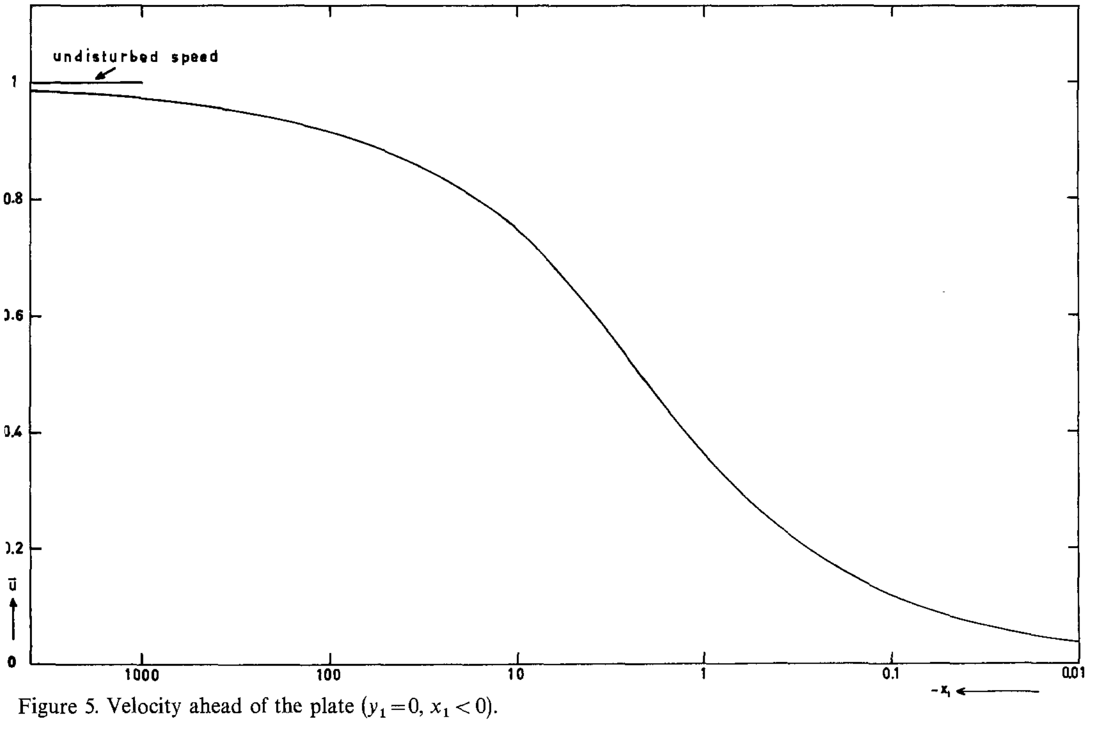

Figure 5. Velocity ahead of the plate ($y_1=0$, $x_1<0$).

TABLE 1

j	$\xi=\sqrt{R_x}$	${}^{10}\log R_x$	$R_x c_f$ (eq. 8.1)	$4I$ (eq. 8.3)	η^* (eq. 8.6)	$\bar{p}-\bar{p}_\infty$ (eq. 8.8)
0	0	$-\infty$	0.7549	0		
2	0.2725	-1.129	0.7571	0.0500	0.682	1.16×10^{-1}
4	0.5974	-0.447	0.7616	0.1119	0.698	1.02×10^{-1}
6	0.9891	-0.009	0.7655	0.1900	0.722	8.56×10^{-2}
8	1.4672	0.333	0.7664	0.2878	0.748	6.87×10^{-2}
10	2.0591	0.627	0.7631	0.4073	0.771	5.29×10^{-2}
12	2.8045	0.896	0.7554	0.5494	0.788	3.91×10^{-2}
14	3.7627	1.151	0.7441	0.7136	0.799	2.76×10^{-2}
16	5.0257	1.402	0.7302	0.8976	0.806	1.84×10^{-2}
18	6.7437	1.658	0.7153	1.0979	0.815	1.15×10^{-2}
20	9.1772	1.925	0.7007	1.3089	0.826	6.58×10^{-3}
22	12.817	2.216	0.6877	1.5231	0.839	3.36×10^{-3}
24	18.693	2.543	0.6773	1.7311	0.849	1.48×10^{-3}
26	29.358	2.935	0.6702	1.9230	0.856	5.41×10^{-4}
28	53.038	3.449	0.6662	2.0918	0.859	1.48×10^{-4}
30	135.34	4.263	0.6645	2.2350	0.860	1.94×10^{-5}
32	∞	∞	0.6641	2.3428	0.860	0

TABLE 2
($\xi=\sqrt{R_x}=135.34$; $\tilde{K}_1$ is the asymptotic approximation (5.4) with $C_2=-2$ (see also sec. 8.3)).

k	η	$\Psi_1(\xi,\eta)$ (sec. 6)	C_2 (eq. 8.5)	$K_1(\xi,\eta)$ (sec. 6)	$\tilde{K}_1(\xi,\eta)$ (eq. 5.4)	$\bar{u}(0,\eta)$ (eq. 8.10)
0	0	0		0.02505	0.02488	0
4	0.2158	0.00231	-2.14	0.02358	0.02350	0.0814
8	0.6055	0.01732	-2.13	0.01730	0.01742	0.2257
12	1.1400	0.05310	-2.11	-0.00166	-0.00119	0.4041
16	1.7708	0.09199	-2.05	-0.01631	-0.01604	0.5638
20	2.4402	0.10853	-1.98	-0.00737	-0.00779	0.6725
24	3.1055	0.11104	-1.95	-0.00089	-0.00110	0.7394
28	3.8139	0.11117	-1.96	-0.00002	-0.00004	0.7859
32	5.0000	0.11107	-1.98	0		0.8349
40	6.3557	0.11084	-1.99			0.8690
48	9.5691	0.11003	-2.01			0.9120
56	20.858	0.10635	-2.05			0.9591
64	∞	0				1

9. Appendix

The behaviour of Ψ_1 and K_1 for large values of ξ.

Eq. (5.4) give the first terms of the asymptotic series of Ψ_1 and K_1 for large ξ. They contain the functions $h(2\eta)$ and $g(2\eta)$, which themselves satisfy differential equations. Substitute eqs. (5.4) into the first equation (5.2) and consider the terms $O(\xi^{-1}\ln\xi)$. In order that these vanish, we should have

$$2h''''+fh'''+3f'h''+f''h'-f'''h=0\,,$$

with boundary conditions

$$\eta=0 \quad h=h'=0$$
$$\eta\to\infty \quad h'\to 0, \text{ exponentially.}$$

The differential equation can be integrated once with the result that

$$2h'''+fh''+2f'h'-f''h=0\,, \tag{A.1}$$

where the integration constant has been put equal to 0, since for $\eta\to\infty$ we must have f'', h', h'', $h'''\to 0$.

Using the fact that the Blasius function f is an integrating factor of (A.1) we obtain

$$2fh''+(f^2-2f')h'+2f''h=0\,,$$

where the integration constant vanishes again.

The general solution of this second order equation is

$$h(2\eta)=C_1(f-2\eta f')+C_2 f'\,.$$

The boundary condition $h'(0)=0$ yields $C_2=0$ and the solution which satisfies the boundary conditions is

$$h(2\eta)=C_1(f-2\eta f')$$

where C_1 is for the moment still arbitrary. The function h is seen to be identical to the first eigensolution of the first order boundary layer equations.

We now consider the terms of $O(\xi^{-1})$ in the first equation (5.2) after substitution of eqs. (5.4). These yield

$$2g''''+fg'''+3f'g''+f''g'-f'''g=4\eta f'''+2\eta^2 f'f''+\eta ff''+f'h''-f'''h\,.$$

Substituting the value obtained for h, we have

$$f'h''-f'''h=-C_1(ff'')'\,.$$

Also there holds

$$4\eta f'''+2\eta^2 f'f''+\eta ff''=\tfrac{1}{4}\frac{d}{d(2\eta)}(f-2\eta f')^2\,.$$

With aid of the two last relations we can integrate the differential equation to

$$2g'''+fg''+2f'g'-f''g=\tfrac{1}{4}\{(f-2\eta f')^2-\beta^2\}-C_1 ff''\,, \tag{A.2}$$

where the integration constant $-\frac{1}{4}\beta^2$ has been obtained from the requirement that $g'\to 0$ for $\eta\to\infty$.

The solution of equation (A.2) is

$$g(2\eta)=C_2(f-2\eta f')+\Delta g\,,$$

where C_2 is an arbitrary constant and Δg is a particular solution satisfying the boundary conditions

$$\eta=0 \quad \Delta g=(\Delta g)'=(\Delta g)''=0$$
$$\eta\to\infty \quad (\Delta g)'\to 0, \quad \text{exponentially}\,.$$

The constant C_1 should have a definite value in order that eq. (A.2) possesses a solution of which all derivatives vanish exponentially at infinity.

This value can be obtained by multiplication of eq. (A.2) with f and integration between o and ∞. The left hand side then becomes

$$2fg''+(f^2-2f')g'+2f''g\Big|_0^\infty$$

which vanishes at both limits if g' and g'' decrease exponentially for $\eta\to\infty$. Hence, the right hand side must also be equal to 0, which gives

$$\int_0^\infty[\tfrac{1}{4}f\{(f-2\eta f')^2-\beta^2\}-C_1 f^2 f'']\,d(2\eta)=0\,.$$

This yields, using (3.4)

$$C_1 = \tfrac{1}{4}\int_0^\infty f\{(f-2\eta f')^2-\beta^2\}\,d(2\eta) = -1.65906126\,.$$

Then $h''(0) = -C_1 f''(0) = 0.55090346$.
Also there holds

$$h(\infty) = -\beta C_1 = 2.85489213$$

$$g(\infty) = -\beta C_2 + \Delta g(\infty)$$

$$= -1.72078765 C_2 - 2.20722510\,.$$

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Note added in proof:

An improved method of numerical solution as well as additional numerical results for the flow field will be published in E. F. F. Botta and D. Dijkstra, *Report* TW-80, Math. Dept., University of Groningen (1970).